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Bianchi Type I Magnetized Barotropic Perfect Fluid Cosmological Model in General Relativity

Raj Bali · Rajni Gupta · Radhika Goyal

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Abstract Bianchi Type I barotropic perfect fluid cosmological model in presence of magnetic field, is investigated. To get the deterministic model, we have also assumed that $\sigma_1^1 \alpha \theta$ where σ_1^1 is the eigen-value of shear tensor σ_i^j and θ the expansion in the model. The behavior of the model in presence and absence of magnetic field and singularities in the model are also discussed.

Keywords Bianchi I · Barotropic · Magnetized · Perfect fluid

1 Introduction

General Relativity finds an interesting application to an investigation of state in which radiation is concentrated around a star. Friedmann [1] constructed world models which have been described by those solutions of Einstein's field equation with incoherent matter where world lines have vanishing shear and rotation. If expansion vanishes for all values of t, we get Einstein universe and if density vanishes, we obtain de-Sitter universe as a particular case of Friedmann model. Roy and Singh [2] investigated Bianchi Type I non-static plane symmetric space-time filled with disordered radiation of perfect fluid distribution. The evolution of deviation from perfect isotropy is dominated by the distortion created by anisotropic stresses. The anisotropic stresses to the gravitational field arise from magnetic fields, collision less relativistic particles, hydrodynamic shear viscosity, gravitational waves, skew axions fields in low energy string or topological defects. The presence of strong magnetic field raises the interesting problem like the formation of galaxies in the actual universe. Primordial magnetic fields of cosmological origin have been speculated by Asseo and Sol [3]. FRW models are approximately valid as present day magnetic field strength is very small. Thorne [4] investigated an LRS (Locally Rotationally Symmetric) Bianchi Type I cosmological model with magnetic field. Collins [5] gave a qualitative analysis of Bianchi Type I models in the presence of magnetic field. Roy and Prakash [6] investigated Bianchi Type I non-static

R. Bali (✉) · R. Gupta · R. Goyal
Department of Mathematics, University of Rajasthan, Jaipur 302004, India
e-mail: balir5@yahoo.co.in

cosmological model for perfect fluid distribution with magnetic field in which free gravitational field is of Petrov Type I degenerate. Bali [7], Bali and Tyagi [8] have investigated some Bianchi Type I cosmological models in presence of magnetic field assuming the conditions $\sigma_1^1 \alpha \theta$ where σ_1^1 is the eigen values of σ_i^j and θ the expansion in the model and for stiff perfect fluid distribution where σ_i^j is the shear tensor. A cosmological model which contains a global magnetic field is necessarily anisotropic since the magnetic field vector specifies a preferred spatial direction. Bronnikov et al. [9] have studied the evolution of Bianchi Type I space-time with a global unidirectional electromagnetic field. Cosmological models with an incident magnetic field for different Bianchi Types have been investigated by several authors viz. Tupper [10], Dunn and Tupper [11], Lorentz [12], Roy et al. [13], Roy and Singh [14], Ribeiro and Sanyal [15], Nayak and Bhuyan [16], Banerjee et al. [17], Tikekar and Patel [18, 19], Roy and Banerjee [20], Bali and Jain [21], Wang [22], Singh and Chaubey [23], Bali and Jain [24].

In this paper, we have investigated Bianchi Type I barotropic perfect fluid cosmological model in the presence of magnetic field. To get the deterministic model of the universe, we have also assumed that $\sigma_1^1 \alpha \theta$ and σ_1^1 is the eigen value of σ_i^j and θ the expansion in the model. The behavior of the model in the presence and absence of magnetic field and singularities in the model are discussed.

2 The Metric and Field Equations

We consider the Bianchi type I metric in the form

$$ds^2 = -dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \tag{2.1}$$

where A, B, C are functions of t-alone.

The energy momentum tensor for perfect fluid with magnetic field is given by

$$T_i^j = (\epsilon + p) v_i v^j + p g_i^j + E_i^j \tag{2.2}$$

where E_i^j is the electro magnetic field given as Lichnerowicz [25]

$$E_i^j = \bar{\mu}\left[|h|^2\left(v_i v^j + \frac{1}{2} g_i^j\right) - h_i h^j\right] \tag{2.3}$$

with

$$h_i = \frac{\sqrt{-g}}{2\bar{\mu}} \epsilon_{ijk\ell} F^{k\ell} v^j, \qquad |h|^2 = h_\ell h^\ell \tag{2.4}$$

where ϵ is the energy density, p the isotropic pressure, h_i the magnetic flux vector, F_{ij} the electromagnetic field tensor, $\epsilon_{ijk\ell}$ the Levi-Civita symbol, and v^i the flow velocity satisfying

$$g_{ij} v^i v^j = -1.$$

We assume that coordinates to be comoving so that

$$v^1 = 0 = v^2 = v^3, \qquad v^4 = 1.$$

The incident magnetic field is taken along x-axis so that

$$h_1 \neq 0, \qquad h_2 = 0, \qquad h_3 = 0, \qquad h_4 = 0.$$

We assume that F_{23} is the only non-vanishing component of F_{ij}.

Maxwell's equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0$$

and

$$\frac{\partial}{\partial x^j}(F^{ij}\sqrt{-g}) = 0$$

together leads to

$$F_{23} = \text{constant} = H \ (\text{say}).$$

Hence

$$h_1 = \frac{AH}{\bar{\mu}BC}. \tag{2.5}$$

Here $F_{14} = 0 = F_{24} = F_{34}$ due to the assumption of infinite electrical conductivity (Roy Maartens [26]).

Thus (2.3) leads to

$$E_1^1 = -\frac{H^2}{2\bar{\mu}B^2C^2} = -E_2^2 = -E_3^3 = E_4^4. \tag{2.6}$$

The Einstein field equation

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi T_i^j$$

for the metric (2.1) leads to

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} = -8\pi\left(p - \frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{2.7}$$

$$\frac{A_{44}}{A} + \frac{C_{44}}{C} + \frac{A_4C_4}{AC} = -8\pi\left(p + \frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{2.8}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} = -8\pi\left(p + \frac{H^2}{2\bar{\mu}B^2C^2}\right), \tag{2.9}$$

$$\frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} = 8\pi\left(\epsilon + \frac{H^2}{2\bar{\mu}B^2C^2}\right). \tag{2.10}$$

3 Solution of Field Equations

Equations (2.7) and (2.8) lead to

$$\frac{A_{44}}{A} - \frac{B_{44}}{B} + \left(\frac{A_4}{A} - \frac{B_4}{B}\right)\frac{C_4}{C} = -8\pi\left(\frac{H^2}{\bar{\mu}B^2C^2}\right). \tag{3.1}$$

From (2.8) and (2.9), we have

$$\frac{B_{44}}{B} - \frac{C_{44}}{C} + \frac{A_4}{A}\left(\frac{B_4}{B} - \frac{C_4}{C}\right) = 0. \tag{3.2}$$

Equations (2.7) to (2.10) are four equations in five unknowns A, B, C, ϵ and p. To get the determinate model of the universe, we assume that universe is filled with barotropic perfect fluid distribution. This leads to $p=\gamma\epsilon$ where $0\le\gamma\le 1$. Now (2.7) and (2.10) after using barotropic perfect fluid condition ($p=\gamma\epsilon$) lead to

$$\gamma\frac{A_4}{A}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)+(1+\gamma)\frac{B_4C_4}{BC}+\frac{B_{44}}{B}+\frac{C_{44}}{C}=(\gamma+1)\frac{4\pi H^2}{\bar{\mu}B^2C^2}. \tag{3.3}$$

Equation (3.2) leads to

$$\frac{(CB_4-BC_4)_4}{CB_4-BC_4}=-\frac{A_4}{4} \tag{3.4}$$

which leads to

$$C^2\left(\frac{B}{C}\right)_4=\frac{m}{A}. \tag{3.5}$$

Let

$$BC=\mu,\qquad B/C=\nu. \tag{3.6}$$

Thus, we have

$$B^2=\mu\nu,\qquad C^2=\mu/\nu. \tag{3.7}$$

Using (3.6) and (3.7) in (3.5), we have

$$\frac{\nu_4}{\nu}=\frac{m}{A\mu}. \tag{3.8}$$

The condition $\sigma_1^1\alpha\theta$ leads to

$$A=\ell(BC)^n \tag{3.9}$$

where σ_1^1 is the eigen value of shear tensor σ_i^j and θ the expansion in the model, ℓ the proportionality constant.

Using (3.6), (3.9) in (3.3), we have

$$\frac{\mu_{44}}{\mu}+\left\{\frac{(4n+1)\gamma-1}{4}\right\}\frac{\mu_4^2}{\mu^2}+\left(\frac{1-\gamma}{4}\right)\frac{\nu_4^2}{\nu^2}=\frac{(\gamma+1)K}{\mu^2} \tag{3.10}$$

where

$$K=\frac{4\pi H^2}{\bar{\mu}}. \tag{3.11}$$

Again using (3.8) and (3.9) in (3.10), we have

$$\mu_{44}+\left\{\frac{(4n+1)\gamma-1}{4}\right\}\frac{\mu_4^2}{\mu}+\left(\frac{1-\gamma}{4}\right)\frac{m^2}{\ell^2\mu^{2n+1}}=\frac{(\gamma+1)K}{\mu} \tag{3.12}$$

which leads to

$$f^2=\frac{4(\gamma+1)K}{(4n+1)\gamma-1}+\frac{m^2}{\ell^2(4n+1)}\mu^{-2n}+\alpha$$

where $\mu_4 = f(\mu)$ and $\mu_{44} = ff'$, $f' = \frac{df}{d\mu}$ and α is constant of integration which leads to

$$f = [M + N\mu^{-2n}]^{1/2} \tag{3.13}$$

where

$$M = \frac{4(\gamma+1)K}{[(4n+1)\gamma - 1]} + \alpha \tag{3.14}$$

and

$$N = \frac{m^2}{\ell^2(4n+1)}. \tag{3.15}$$

Equations (3.8) and (3.13) lead to

$$\nu = L\left[\frac{1}{\mu^n} + \sqrt{\frac{1}{\mu^{2n}} + \frac{M}{N}}\right]^{-\frac{m}{n\ell\sqrt{N}}}. \tag{3.16}$$

Thus we have

$$A^2 = \ell^2\mu^{2n}, \tag{3.17}$$

$$B^2 = L\mu\left[\frac{1}{\mu^n} + \sqrt{\frac{1}{\mu^{2n}} + \frac{M}{N}}\right]^{-\frac{m}{n\ell\sqrt{N}}}, \tag{3.18}$$

$$C^2 = \frac{\mu}{L}\left[\frac{1}{\mu^n} + \sqrt{\frac{1}{\mu^{2n}} + \frac{M}{N}}\right]^{\frac{m}{n\ell\sqrt{N}}}. \tag{3.19}$$

Hence the metric (2.1) leads to

$$ds^2 = -\left(\frac{dt}{d\mu}\right)^2 d\mu^2 + \ell^2\mu^{2n}dx^2 + L\mu\left[\frac{1}{\mu^n} + \sqrt{\frac{1}{\mu^{2n}} + \frac{M}{N}}\right]^{-\frac{m}{n\ell\sqrt{N}}} dy^2 + \frac{\mu}{L}\left[\frac{1}{\mu^n} + \sqrt{\frac{1}{\mu^{2n}} + \frac{M}{N}}\right]^{\frac{m}{n\ell\sqrt{N}}} dz^2 \tag{3.20}$$

$$= -\frac{T^{2n}dT^2}{(MT^{2n}+N)} + \ell^2T^{2n}dX^2 + T\left[\frac{1}{T^n} + \sqrt{\frac{1}{T^{2n}} + \frac{M}{N}}\right]^{-\frac{m}{n\ell\sqrt{N}}} dY^2 + T\left[\frac{1}{T^n} + \sqrt{\frac{1}{T^{2n}} + \frac{M}{N}}\right]^{\frac{m}{n\ell\sqrt{N}}} dZ^2 \tag{3.21}$$

where $\mu = T$, $x = X$, $\sqrt{L}y = Y$, $\frac{1}{\sqrt{L}}z = Z$.

In the absence of magnetic field i.e. when $M \to \alpha$ i.e. when $K \to 0$ then the metric (3.21) reduces to

$$ds^2 = -\frac{T^{2n}dT^2}{(\alpha T^{2n}+N)} + \ell^2T^{2n}dX^2 + T\left[\frac{1}{T^n} + \sqrt{\frac{1}{T^{2n}} + \frac{\alpha}{N}}\right]^{-\frac{m}{n\ell\sqrt{N}}} dY^2 + T\left[\frac{1}{T^n} + \sqrt{\frac{1}{T^{2n}} + \frac{\alpha}{N}}\right]^{\frac{m}{n\ell\sqrt{N}}} dZ^2. \tag{3.22}$$

4 Some Physical and Geometrical Features

The energy density (ϵ), the isotropic pressure (p) for the model (3.21) are given by

$$8\pi\epsilon = \frac{1}{T^2}\left[\frac{M(4n+1)}{4} - \frac{\{(4n+1)\gamma+1\}K}{(4n+1)\gamma-1}\right] \tag{4.1}$$

and

$$8\pi p = \frac{\gamma}{T^2}\left[\frac{M(4n+1)}{4} - \frac{\{(4n+1)\gamma+1\}K}{(4n+1)\gamma-1}\right]. \tag{4.2}$$

The expansion (θ) and components of shear tensor (σ_i^j) and conformal curvature tensor (C_{hi}^{jk}) are given by

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} = \frac{(n+1)}{T^{n+1}}[MT^{2n}+N]^{1/2}, \tag{4.3}$$

$$\sigma_1^1 = \frac{1}{3}\left(\frac{2A_4}{A} - \frac{B_4}{B} - \frac{C_4}{C}\right) = \left(\frac{2n-1}{3}\right)\frac{(MT^{2n}+N)^{1/2}}{T^{n+1}}, \tag{4.4}$$

$$\sigma_2^2 = \frac{1}{3}\left(\frac{2B_4}{B} - \frac{A_4}{A} - \frac{C_4}{C}\right) = \frac{1}{3T^{n+1}}\left[\left(\frac{1-2n}{2}\right)(MT^{2n}+N)^{1/2} + \frac{3m}{2\ell}\right], \tag{4.5}$$

$$\sigma_3^3 = \frac{1}{3}\left(\frac{2C_4}{C} - \frac{A_4}{A} - \frac{B_4}{B}\right) = \frac{1}{3T^{n+1}}\left[\left(\frac{1-2n}{2}\right)(MT^{2n}+N)^{1/2} - \frac{3m}{2\ell}\right], \tag{4.6}$$

$$\sigma_4^4 = 0.$$

Thus

$$\sigma^2 = \frac{1}{2}[(\sigma_1^1)^2 + (\sigma_2^2)^2 + (\sigma_3^3)^2 + (\sigma_4^4)^2].$$

Therefore

$$\sigma = \frac{1}{2\sqrt{3}T^{n+1}}\left[(1-2n)^2(MT^{2n}+N) + \frac{3m^2}{\ell^2}\right]^{1/2}, \tag{4.7}$$

$$C_{12}^{12} = -\frac{1}{6T}\left[\frac{3nm^2}{(4n+1)\ell^2T^{2n}} - \frac{(2n^2-3n+1)}{2}M + \frac{3mn}{\ell}\frac{(MT^{2n}+N)^{1/2}}{T^{2n}}\right], \tag{4.8}$$

$$C_{13}^{13} = -\frac{1}{6T}\left[\frac{3nm^2}{(4n+1)\ell^2T^{2n}} - \frac{(2n^2-3n+1)}{2}M - \frac{3mn}{\ell}\frac{(MT^{2n}+N)^{1/2}}{T^{2n}}\right], \tag{4.9}$$

$$C_{14}^{14} = \frac{1}{6T}\left[\frac{6nm^2}{(4n+1)\ell^2T^{2n}} - (2n^2-3n+1)M\right]. \tag{4.10}$$

The spatial volume (R^3) and the deceleration parameter (q) are given by

$$R^3 = \ell T^{n+1}, \tag{4.11}$$

$$q = -\frac{[(n-2)M - \frac{2(n+1)N}{T^{2n}}]}{(n+1)[M + \frac{N}{T^{2n}}]}. \tag{4.12}$$

5 Discussion

The reality conditions (i) $\epsilon + p > 0$ (ii) $\epsilon + 3p > 0$ given by Ellis [27] together lead to

$$\frac{M(4n+1)}{4} > \frac{\{(4n+1)\gamma + 1\}K}{(4n+1)\gamma - 1}. \tag{5.1}$$

The model (3.21) in the presence of magnetic field exists during the span of time $T = 0$ to $T = \infty$ having restrictions on constant quantities given by (5.1). The model (3.21) starts with a big-bang at $T = 0$ and the expansion in the model decreases as time increases. Since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$, hence the model does not isotropize for large values of T. Since $q < 0$ for $T^{2n} > \frac{2(n+1)N}{(n-2)M}$, hence the model (3.21) represents an accelerating universe. However $q > 0$ for $T^{2n} < \frac{2(n+1)N}{(n-2)M}$, hence it represents decelerating universe during this span of time. The matter density $\epsilon \to \infty$ when $T \to 0$ and $\epsilon \to 0$ when $T \to \infty$ in the presence of magnetic field. The model in general represents non-rotating and Petrov Type I degenerate in the presence of magnetic field. The model (3.21) has Point Type singularity (MacCallum [28]) at $T = 0$ when $n > 0$ and has Cigar Type singularity at $T = 0$ when $n < 0$.

In the absence of magnetic field, the above mentioned quantities are given by

$$8\pi\epsilon = \frac{(4n+1)\alpha}{4T^2}, \tag{5.2}$$

$$8\pi p = \frac{(4n+1)\gamma\alpha}{4T^2}, \tag{5.3}$$

$$\theta = \frac{(n+1)m}{T}\left[\frac{m^2}{\ell^2(4n+1)T^{2n}} + \alpha\right]^{1/2}, \tag{5.4}$$

$$\sigma = \frac{1}{\sqrt{3}}\frac{1}{T}\left[\frac{(1-2n)^2\alpha}{4} + \frac{(n+1)^2m^2}{(4n+1)\ell^2T^{2n}}\right]^{1/2}. \tag{5.5}$$

The components of conformal curvature tensor, the spatial volume and the deceleration parameter in the absence of magnetic field are given by

$$C^{12}_{12} = -\frac{1}{6T}\left[\frac{3nm^2}{(4n+1)\ell^2T^{2n}} - \frac{(2n^2-3n+1)}{2}\alpha + \frac{3mn}{\ell}\frac{(\alpha T^{2n}+N)^{1/2}}{T^{2n}}\right], \tag{5.6}$$

$$C^{13}_{13} = -\frac{1}{6T}\left[\frac{3nm^2}{(4n+1)\ell^2T^{2n}} - \frac{(2n^2-3n+1)}{2}\alpha - \frac{3mn}{\ell}\frac{(\alpha T^{2n}+N)^{1/2}}{T^{2n}}\right], \tag{5.7}$$

$$C^{14}_{14} = \frac{1}{6T}\left[\frac{6nm^2}{(4n+1)\ell^2T^{2n}} - (2n^2-3n+1)\alpha\right], \tag{5.8}$$

$$R^3 = \ell T^{n+1}, \tag{5.9}$$

$$q = -\frac{[(n-2)\alpha - \frac{2(n+1)N}{T^{2n}}]}{(n+1)[\alpha + \frac{N}{T^{2n}}]}. \tag{5.10}$$

In the absence of magnetic field, the reality conditions (i) $\epsilon + p > 0$, (ii) $\epsilon + 3p > 0$ together lead to $(4n+1)\alpha > 0$. The model in the absence of magnetic field also starts with a big-bang at $T = 0$ and the expansion in the model decreases as time increases. The spatial volume increases as time increases when $n+1 > 0$. Since $q < 0$ for $T^{2n} > \frac{2(n+1)N}{(n-2)\alpha}$, hence the model (3.22) represents an accelerating universe. However if $T^{2n} < \frac{2(n+1)N}{(n-2)\alpha}$ then the model (3.22) represents a decelerating universe. The model (3.22) has Point Type singularity at $T = 0$ when $n > 0$ and Cigar Type singularity at $T = 0$ when $n < 0$. The model represents Petrov Type I degenerate in absence of magnetic field. For large values of T, the model also represents a conformally flat space-time. When $T \to 0$ then $\epsilon \to \infty$ and $\epsilon \to 0$ when $T \to \infty$. Since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$. Hence the anisotropy is maintained for large values of T.

References

1. Friedmann, A.: Z. Phys. **10**, 377 (1922). doi:10.1007/BF01332580
2. Roy, S.R., Singh, P.N.: J. Phys. A : Math. Gen. 10, 1, 49 (U.K.) (1977)
3. Asseo, E., Sol, H.: Phys. Rep. **6**, 148 (1987)
4. Thorne, K.S.: Astrophys. J. **148**, 51 (1967). doi:10.1086/149127
5. Collins, C.B.: Commun. Math. Phys. **27**, 37 (1972). doi:10.1007/BF01649657
6. Roy, S.R., Prakash, S.: Ind. J. Phys. B. **52**, 47 (1978)
7. Bali, R.: Int. J. Theor. Phys. USA **25**, 7 (1986). doi:10.1007/BF00669712
8. Bali, R., Tyagi, A.: Int. J. Theor. Phys. USA **27**, 5 (1988)
9. Bronnikov, K.A., Chudayeva, E.N., Shikin, G.N.: Class. Quantum Gravity **21**, 3389 (2004). doi:10.1088/0264-9381/21/14/005
10. Tupper, B.O.J.: Phys. Rev. D Part. Fields **15**, 2123 (1977). doi:10.1103/PhysRevD.15.2123
11. Dunn, K.A., Tupper, B.O.J.: Astrophys. J. **235**, 307 (1980). doi:10.1086/157635
12. Lorentz, D.: Lett. Nuovo Cimento B **29**, 238 (1980). doi:10.1007/BF02743319
13. Roy, S.R., Narain, S., Singh, J.P.: Aust. J. Phys. **38**, 239 (1985)
14. Roy, S.R., Singh, J.P.: Aust. J. Phys. (1985)
15. Ribeiro, M.B., Sanyal, A.K.: J. Math. Phys. **28**, 657 (1987). doi:10.1063/1.527599
16. Nayak, B.K., Bhuyan, G.B.: Gen. Relativ. Gravit. **19**, 939 (1987). doi:10.1007/BF00759298
17. Banerjee, A., Sanyal, A.K., Chakraborty, S.C.: Pramana J. Phys. **34**, 1 (1990)
18. Tikekar, R., Patel, L.K.: Gen. Relativ. Gravit. **24**, 397 (1992). doi:10.1007/BF00760415
19. Tikekar, R., Patel, L.K.: Pramana J. Phys. **42**, 483 (1994)
20. Roy, S.R., Banerjee, S.K.: Class. Quantum Gravity 2845 (1997)
21. Bali, R., Jain, V.C.: Astrophys. Space Sci. **262**, 145 (1999)
22. Wang, X.X.: Chin. Phys. Lett. **23**, 1702 (2006). doi:10.1088/0256-307X/23/11/012
23. Singh, T., Chaubey, R.: Int. J. Mod. Phys. D **15**, 493 (2006). doi:10.1142/S0218271806008334
24. Bali, R., Jain, S.: Int. J. Mod. Phys. D **16**, 11 (2007). doi:10.1142/S0218271807011073
25. Lichnerowicz, A.: Relativistic Hydrodynamics and Magnetohydro-Dynamics, p. 13. Benjamin, New York (1967),
26. Roy, M.: Pramana J. Phys. **55**, 575 (2000)
27. Ellis, G.F.R.: In: Sachs, R.K. (ed.) General Relativity and Cosmology. Clarendon, London (1971)
28. McCallum, M.A.H.: Commun. Math. Phys. **20**, 57 (1971). doi:10.1007/BF01646733